MTH 166

Lecture-7

Linear Differential Equations (LDE)

Unit 2: **Differential Equations of Higher Order**

(**Book:** Advanced Engineering Mathematics by R.K.Jain and S.R.K Iyengar,**Chapter-5**)

Topic:

Linear Differential Equations (LDE)

Learning Outcomes:

1. Identification of Linear Differential Equations (LDE).

2. Necessary and Sufficient condition for LDE to be Normal on an interval

3. Show functions as solutions of LDE.

Linear Differential Equations (LDE):

A linear differential equation of order n is written as:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + --- + a_{n-1} \frac{dy}{dx} + a_n y = r(x) \qquad (1)$$

or

$$a_0 y^n + a_1 y^{n-1} + ---- + a_{n-1} y' + a_n y = r(x) \qquad (1)$$

For example, a second order LDE is written as:

$$a_0 y'' + a_1 y' + a_2 y = r(x) \qquad (2)$$

***** If $r(x) = 0$, then LDE is called Homogeneous LDE.

***** If $r(x) \neq 0$, then LDE is called Non-Homogeneous LDE.

***** If $a_0, a_1, ---, a_n$ are all constants, then LDE is called LDE with constant coefficients.

***** If $a_0, a_1, ---, a_n$ are not all constants, then it is called LDE with variable coefficients.

Classify the following LDE:

1. $y'' + 4y' + 3y = x^2 e^x$

It is a 2nd order Non-homogeneous LDE with constant coefficients.

2. $y'' + 2y' + y = \sin x$

It is a 2nd order Non-homogeneous LDE with constant coefficients.

3. $x^2 y'' + xy' + (x^2 - 4)y = 0$

It is a 2nd order Homogeneous LDE with variable coefficients.

4. $(1 - x^2)y'' - 2xy' + 20y = 0$

It is a 2nd order Homogeneous LDE with variable coefficients.

Necessary and Sufficient condition for LDE to be Normal on an interval:

A linear differential equation of order n:

$$a_0 y^n + a_1 y^{n-1} + ---- + a_{n-1} y' + a_n y = r(x) \qquad (1)$$

is said to be normal on an interval I if:

1. $a_0, a_1, ---, a_n$ and $r(x)$ are all continuous on an interval I

2. $a_0 \neq 0$

* In homogeneous LDE ($r(x)$=0), the problem arises only because of $a_0 \neq 0$.

* In non-homogeneous LDE ($r(x) \neq 0$), the problem arises due to $a_0 \neq 0$ and also due to domain of $r(x)$.

* In almost all numerical $a_0, a_1, ---, a_n$ are all continuous, so first condition automatically holds. We are focused on second condition only.

<u>Find the intervals on which the following differential equations are normal.</u>

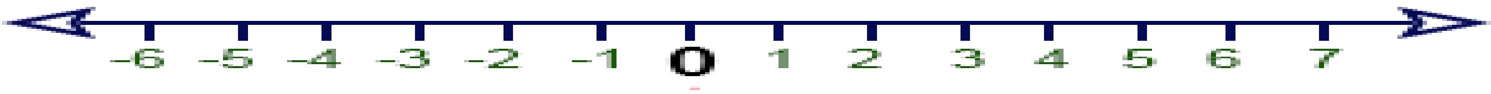
<u>Problem 1.</u> $\boldsymbol{(1-x^2)y''-2xy'+3y=0}$

<u>Solution:</u> $(1-x^2)y''-2xy'+3y=0$ (1)

Comparing with: $a_0y''+a_1y'+a_2y=0$

1. $\left.\begin{array}{l} a_0=(1-x^2) \\ a_1=-2x \\ a_2=3 \end{array}\right\}$ Being polynomials, a_0, a_1, a_2 are all continuous on $(-\infty,\infty)$

2. $a_0 \neq 0$ $\Rightarrow (1-x^2)\neq 0$ $\Rightarrow x^2 \neq 1$ $\Rightarrow x \neq \pm 1$



Thus, LDE (1) is normal on subintervals: $(-\infty,-1),(-1,1),(1,\infty)$.

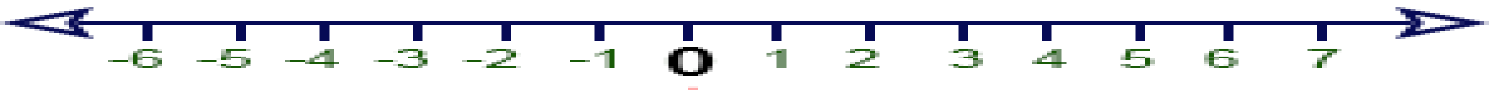
Problem 2. $\boldsymbol{x^2y'' + xy' + \left(n^2 - x^2\right)y = 0}$**; $\boldsymbol{n}$ is real.**

Solution: $x^2y'' + xy' + (n^2 - x^2)y = 0$ (1)

Comparing with: $a_0y'' + a_1y' + a_2y = 0$

1. $\left.\begin{aligned} a_0 &= x^2 \\ a_1 &= x \\ a_2 &= (n^2 - x^2) \end{aligned}\right\}$ Being polynomials, a_0, a_1, a_2 are all continuous on $(-\infty, \infty)$

2. $a_0 \neq 0 \qquad \Rightarrow x^2 \neq 0 \qquad \Rightarrow x \neq 0$



Thus, LDE (1) is normal on subintervals: $(-\infty, 0), (0, \infty)$.

Problem 3. $\boldsymbol{y'' + 9y' + y = \log\left(x^2 - 9\right)}$

Solution: $y'' + 9y' + y = \log(x^2 - 9)$ (1)

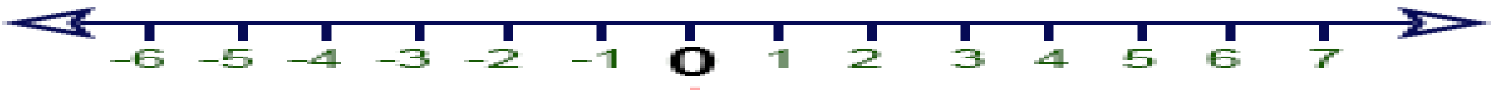
Comparing with: $a_0 y'' + a_1 y' + a_2 y = r(x)$

1. $\left.\begin{aligned} a_0 &= 1 \\ a_1 &= 9 \\ a_2 &= 1 \end{aligned}\right\}$ Being constants, a_0, a_1, a_2 are all continuous on $(-\infty, \infty)$

Also $r(x) = \log(x^2 - 9)$ will be defined if $(x^2 - 9) > 0$ i.e. $x^2 > 9$

$\Rightarrow |x| > 3 \qquad \Rightarrow -\infty < x < -3$ and $3 < x < \infty$

2. $a_0 \neq 0 \qquad \Rightarrow 1 \neq 0$ which is true.



Thus, LDE (1) is normal on subintervals: $(-\infty, -3), (3, \infty)$.

Problem 4. $\boldsymbol{\sqrt{x}y'' + 6xy' + 15y = \log\left(x^4 - 256\right)}$

Solution: $\sqrt{x}y'' + 6xy' + 15y = \log(x^4 - 256)$ (1)

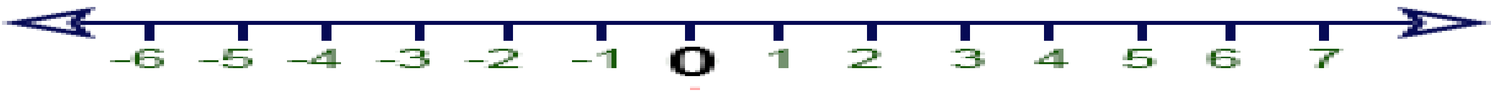
Comparing with: $a_0y'' + a_1y' + a_2y = r(x)$

1. $\left.\begin{aligned} a_0 &= \sqrt{x} \\ a_1 &= 6x \\ a_2 &= 15 \end{aligned}\right\}$ Being polynomials, a_0, a_1, a_2 are all continuous on $(-\infty, \infty)$

Also $r(x) = \log(x^4 - 256)$ will be defined if $(x^4 - 256) \geq 0$ i.e. $x^2 \geq 9$

$\Rightarrow |x| \geq 3 \qquad \Rightarrow -\infty < x \leq -3$ and $3 \leq x \leq \infty$

2. $a_0 \neq 0 \qquad \Rightarrow \sqrt{x} \neq 0 \Rightarrow x > 0$ (Square root of a negative number is not defined)



Thus, LDE (1) is normal on subintervals: $(3, \infty)$.

Principle of Superposition:

If functions $(y_1, y_2, ---, y_n)$ are the solutions of homogeneous LDE:

$$a_0 y^n + a_1 y^{n-1} + ----- + a_{n-1} y' + a_n y = 0 \qquad (1)$$

Then, their linear combination: $(c_1 y_1 + c_2 y_2 + --- + c_n y_n)$ is also a solution of LDE (1).

Note: Principle of superposition is not applicable to non-homogeneous LDE.

Problem: Show that (e^x, e^{-x}) and their linear combination $(c_1e^x + c_2e^{-x})$ are the solutions of homogeneous equation: $y'' - y = 0$

Solution: The given homogeneous LDE: $y'' - y = 0$ (1)

Let $(y_1, y_2) = (e^x, e^{-x})$ be the given set of functions.

Now they will be solutions of equation (1) if they satisfy equation (1).

i.e. y_1 will be solution of equation (1) if $y_1'' - y_1 = 0$ (2)

Here $y_1 = e^x$ $\Rightarrow y_1' = e^x$ $\Rightarrow y_1'' = e^x$

Substitute these values of y_1 and y_1'' in equation (2), we get:

$e^x - e^x = 0$ which is true.

So, y_1 is a solution of equation (1)

Now y_2 will be solution of equation (1) if $y_2'' - y_2 = 0$ (3)

Here $y_2 = e^{-x}$ $\Rightarrow y_2' = -e^{-x}$ $\Rightarrow y_2'' = e^{x}$

Substitute these values of y_2 and y_2'' in equation (3), we get:

$e^{x} - e^{x} = 0$ which is true.

So, y_2 is a solution of equation (1)

By principle of superposition, if y_1 and y_2 are solutions, then their linear combination $(c_1e^{x} + c_2e^{-x})$ will also be a solution.

Let us verify it:

Let $y_3 = (c_1e^{x} + c_2e^{-x})$ $\Rightarrow y_3' = (c_1e^{x} - c_2e^{-x}) \Rightarrow y_3'' = (c_1e^{x} + c_2e^{-x})$

Now y_3 will be solution of equation (1) if $y_3'' - y_3 = 0$

i.e. $(c_1e^{x} + c_2e^{-x}) - (c_1e^{x} + c_2e^{-x}) = 0$ which is true.